Final - Analysis of Several Variables (2023-24)
Time: 3 hours.

Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof, but not results on differential forms.

1. (a) Give an example of a scalar field $f(x,y)$ on $\mathbf{R}^2$ which has partial derivatives everywhere, yet is not continuous at at least one point. **[3 marks]**

   (b) If a scalar field $f(x,y)$ on $\mathbf{R}^2$ has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ everywhere in an open set S, and if there is an $M > 0$ such that

$$\left|\frac{\partial f}{\partial x}(x,y)\right| < M, \qquad \left|\frac{\partial f}{\partial y}(x,y)\right| < M \qquad \text{for all } (x,y) \in S,$$

   then show that f is continuous everywhere in S. **[4 marks]**

2. (a) Consider two bounded regions $S, T \subset \mathbf{R}^2$. Denote points in T by (u,v) and points in S by (x,y). Suppose there is a 1-1 map from T onto S given by $x = X(u,v)$, $y = Y(u,v)$. The change of variables formula gives

$$\iint_S f(x,y)\, dxdy = \iint_T f\left[X(u,v), Y(u,v)\right] |J(u,v)| dudv,$$

   for continuous functions f on S. Derive this formula *non-rigourously* by looking at small rectangular regions in T; in particular show how the factor $|J(u,v)|$ arises in the right hand side. (Do NOT give the rigourous proof). **[3 marks]**

   (b) Next let $S = \mathbf{r}(T)$ be a parametric surface described by a differentiable function $\mathbf{r}$ defined on a region T in the uv-plane, and let f be a scalar field defined and bounded on S. Define the surface integral $\iint_{\mathbf{r}(T)} f dS$ of f over S, and explain how this arises by looking at small rectangular regions in T as above. **[2 marks]**

3. Compute the area of that portion of the conical surface $x^2 + y^2 = z^2$ which lies between the two planes $z = 0$ and $x + 2z = 3$. **[5 marks]**

4. Show that the vector field

$$\mathbf{f}(x,y) = \left[\sin(xy) + xy\cos(xy)\right]\mathbf{i} + \left[x^2\cos(xy)\right]\mathbf{j}$$

   on $\mathbf{R}^2$ is the gradient of a scalar field, and find a corresponding potential function φ. **[5 marks]**

5. Consider the surface in $\mathbf{r} : [0,1]^2 \to \mathbf{R}^3$ given by $\mathbf{r}(u,v) = (u+v)\mathbf{i} + (u-v)\mathbf{j} + 4v^2\mathbf{k}$.

   (a) Let $\boldsymbol{\alpha}(t)$ be a curve in $[0,1]^2$ so that $\mathbf{r}(\boldsymbol{\alpha}(t))$ is a curve on the surface. Show that the fundamental vector product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is normal to this curve. **[3 marks]**

   (b) Find the tangent plane to the surface at the point corresponding to $(u,v) = (0.5, 0.5)$. **[3 marks]**

6. State Stokes' theorem in $\mathbf{R}^3$ which relates a surface integral to a line integral. **[3 marks]**

7. Let f be a function defined on the rectangle $Q = [a,b] \times [c,d]$. Prove or disprove:

   (a) Prove or disprove: If $|f|$ is integrable on Q then f is integrable on Q. **[3 marks]**

   (b) Prove or disprove: If f is integrable on Q then $|f|$ is integrable on Q. **[3 marks]**

   (c) If both f and $|f|$ are integrable on Q then show that

$$\left|\iint_Q f dxdy\right| \le \iint_Q |f| dxdy. \qquad \textbf{[2 marks]}$$

8. Find

$$\oint_C \frac{-ydx + xdy}{x^2 + y^2}$$

where C is a closed curve in $\mathbf{R}^2$ which goes once around the origin in the counterclockwise direction. **[5 marks]**

9. Let $P, Q : \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable functions, and let Ω be an open set in $\mathbf{R}^2$ with piecewise smooth boundary $\partial\Omega$ having orientation in the *counterclockwise* direction. A two dimensional version of the divergence theorem would state

$$\int_\Omega \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right] dxdy = \int_{\partial\Omega} \left[(P, Q) \cdot (n_1, n_2) \right] ds, \qquad \textbf{(*)}$$

where (n_1, n_2) is the *outward* pointing unit normal vector at a point on $\partial\Omega$, and the integral on the right is the line integral with respect to the arc length. The term $(P, Q) \cdot (n_1, n_2)$ is just the dot product of the two vectors.
Show that **(*)** is exactly Green's theorem. **[6 marks]**